

Parametric Oscillatory Instability in Fabry-Perot (FP) Interferometer

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 (February 7, 2008)

We present an approximate analysis of a nonlinear effect of parametric oscillatory instability in FP interferometer. The basis for this effect is the excitation of the additional (Stokes) optical mode with frequency ω_1 and of the mirror's elastic mode with frequency ω_m when the optical energy stored in the main FP resonator mode with frequency ω_0 exceeds the certain threshold and the frequencies are related as $\omega_0 \simeq \omega_1 + \omega_m$. This effect is undesirable in laser gravitational wave antennae because it may create a specific upper limit for the value of energy stored in FP resonator. In order to avoid it the detailed analysis of the mirror's elastic modes and FP resonator optical modes structure is necessary.

I. INTRODUCTION

The full scale terrestrial gravitational wave antennae are in process of assembling and tuning at present. One of these antennae (LIGO-I project) sensitivity expressed in terms of the metric perturbation amplitude is projected to achieve soon the level of $h \simeq 1 \times 10^{-21}$ [1,2]. In 2008 the projected level of sensitivity has to be not less than $h \simeq 1 \times 10^{-22}$ [3]. This value is scheduled to achieve by substantial improvement of the test masses (mirrors in the big FP resonator) isolation from different sources of noises and by increasing the optical readout system sensitivity. This increase is expected to be obtained by rising the value of optical energy \mathcal{E}_0 stored in the FP resonator optical mode: $\mathcal{E}_0 > 30$ J (it corresponds to the circulating power W bigger than 1 megawatt). So high values of \mathcal{E}_0 and W may be a source of the nonlinear effects which will prevent from reaching the projected sensitivity of $h \sim 1 \times 10^{-22}$. Authors of this article already described two such effects: photo-thermal shot noise [4] (the random absorption of optical photons in the surface layer of the mirror causes the fluctuating of mirror surface due to nonzero coefficient of thermal expansion) and photo-refractive shot noise [5] (the same random absorption of optical photons causes the fluctuations of the reflected wave phase due to the dependence of refraction index on temperature). In this paper we analyze undesirable effect of parametric instability — another "trap" of pure dynamical nonlinear origin which (being ignored) may cause very substantial decrease of the antennae sensitivity and even may make the antenna unable to work properly.

It is appropriate to remind that nonlinear coupling of elastic and light waves in continuous media produces Mandelstam-Brillouin scattering. It is a clas-

sical parametric effect, however, it is often explained in terms of quantum physics: one quantum $\hbar\omega_0$ of main optical wave transforms into two, *i. e.* $\hbar\omega_1$ in the additional optical wave (Stokes wave: $\omega_1 < \omega_0$) and $\hbar\omega_m$ in the elastic wave so that $\omega_0 = \omega_1 + \omega_m$ (it is Manley-Rowe condition for parametric process). The irradiation into the anti-Stokes wave is also possible ($\omega_1 = \omega_0 + \omega_m$), however, in this case the part of energy is taken from the elastic wave. The physical "mechanism" of this coupling is the dependence of refractive index on density which is modulated by elastic waves. If the main wave power is large enough the stimulated scattering will take place, the amplitudes of elastic and Stokes waves will increase substantially. The physical description is the following: the flux of energy into these waves is so large that before being irradiated from the volume of interaction, the oscillations with frequencies ω_1 and ω_m stimulate each other substantially increasing the power taken from the main wave. Note that stimulated scattering causes irradiation only into Stokes wave because the additional energy pump into elastic wave must take place for radiation into anti-Stokes wave.

In gravitational wave antennae elastic oscillations in FP resonator mirrors will interact with optical ones being coupled parametrically due to the boundary conditions on one hand, and due to the ponderomotive force on the other hand. Two optical modes may play roles of the main and Stokes waves. High quality factors of these modes and of the elastic one will increase the effectiveness of the interaction between them and may give birth to the parametric oscillatory instability which is similar to stimulated Mandelstam-Brillouin effect [6]. This instability may create a specific upper limit for the value of energy \mathcal{E}_0 .

It is worth to note that this effect of parametric instability is a particular case of the more general phenomenon related to the dynamical back action of parametric displacement meter on mechanical oscillator or free mass. This dynamical back action was analyzed and observed more than 30 years ago [7,8]. Usually parametric meter consists of e.m. resonator with high quality factor Q (radiofrequency, microwave or optical ones) and high frequency stability pumping self-sustained oscillator. The displacement of the resonator movable element modulates its eigenfrequency which in its turn produces the modulation of the e.m. oscillations amplitude (it can produce also phase or output power modulations). If the experimentalist attaches a probe mass to the movable element of the meter he is inevitably confronted with the effect of

dynamical back action, the ponderomotive force produces a rigidity and due to finite e.m. relaxation time — a mechanical friction. Both these values may be positive and negative ones. In the case when the negative friction is sufficiently high the behavior of mechanical oscillator and meter becomes oscillatory unstable. This effect was observed and explained for the case when value of mechanical frequency ω_m was substantially smaller than the bandwidth of e.m. resonator [7,8]. In this article we analyze the parametric oscillatory instability in two optical modes of FP resonator and elastic mode of the mirror. In this case the value of ω_m is much larger than the bandwidth of the optical modes.

In section II we present the analysis of this effect for simplified one-dimensional model which permits to obtain approximate estimates for the instability conditions. In section III we present considerations and preliminary estimates for nonsimplified three-dimensional model, and in section IV — the program of necessary mode numerical analysis.

II. SIMPLIFIED ONE-DIMENSIONAL MODEL

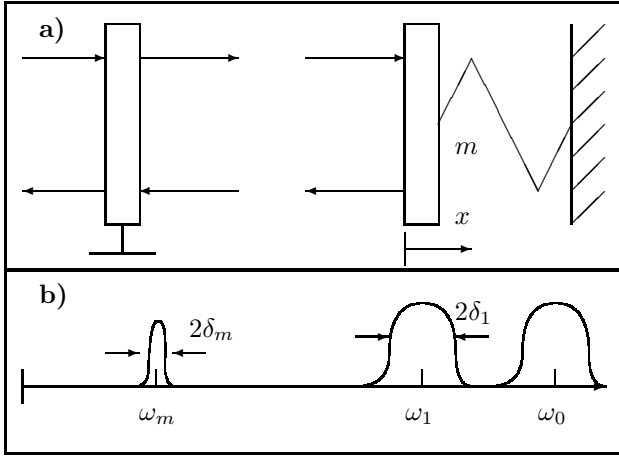


FIG. 1. Scheme of FP resonator with movable mirror (a) and frequency diagram (b).

For approximate estimates we present in this section the simplified model analysis where we assume that:

- The mechanical oscillator (model of mirror) is a lumped one with single mechanical degree of freedom (eigenfrequency ω_m and quality factor $Q_m = \omega_m/2\delta_m$).
- This oscillator mass m is the FP resonator right mirror (see fig. 1) having ideal reflectivity and the value of m is of the order of the total mirror's mass.
- The left mirror (through which FP resonator is pumped) has an infinite mass, no optical losses

and finite transmittance $T = 2\pi L/(\lambda_0 Q_{opt})$ (λ_0 is the optical wavelength, Q_{opt} is the quality factor, L is the distance between the mirrors).

- We take into account only the main mode with frequency ω_0 and relaxation rate $\delta_0 = \omega_0/2Q_0$ and Stokes mode with ω_1 and $\delta_1 = \omega_0/2Q_1$ correspondingly (Q_0 and Q_1 are the quality factors), $\omega_0 - \omega_1 \simeq \omega_m$.
- Laser is pumping only the main mode which stored energy \mathcal{E}_0 is assumed to be a constant one (approximation of constant field).

It is possible to calculate at what level of energy \mathcal{E}_0 the Stokes mode and mechanical oscillator becomes unstable. The origin of this instability can be described qualitatively in the following way: small mechanical oscillations with the resonance frequency ω_m modulate the distance L that causes the exitation of optical fields with frequencies $\omega_0 \pm \omega_m$. Therefore, the Stokes mode amplitude will rise linearly in time if time interval is shorter than δ_1^{-1} . The presence of two optical fields with frequencies ω_0 and ω_1 will produce the component of ponderomotive force (which is proportional to square of sum field) on difference frequency $\omega_0 - \omega_1$. Thus this force will increase the initially small amplitude of mechanical oscillations. In other words, we have to use two equations for Stokes mode and mechanical oscillator and find the conditions when this "feedback" prevails the damping which exists due to the finite values of Q_m and Q_1 . Below we present only the scheme of calculations (see details in Appendix A).

We write down the field components of optical modes and the displacement x of mechanical oscillator in rotating wave approximation:

$$\begin{aligned} E_0 &= A_0[D_0 e^{-i\omega_0 t} + D_0^* e^{i\omega_0 t}], \\ E_1 &= A_1[D_1 e^{-i\omega_1 t} + D_1^* e^{i\omega_1 t}], \\ x &= X e^{-i\omega_m t} + X^* e^{i\omega_m t}, \end{aligned}$$

where D_0 and D_1 are the slowly changing complex amplitudes of the main and Stokes modes correspondingly and X is the slowly changing complex amplitude of mechanical displacement. Normalizing constants A_0 , A_1 are chosen so that energies $\mathcal{E}_{0,1}$ stored in each mode are equal to $\mathcal{E}_{0,1} = \omega_{0,1}^2 |D_{0,1}|^2 / 2$. Then it is easy to obtain the equations for slowly changing amplitudes:

$$\partial_t D_1 + \delta_1 D_1 = \frac{i X^* D_0 \omega_0}{L} e^{-i\Delta\omega t}, \quad (1)$$

$$\partial_t X + \delta_m X = \frac{i D_0 D_1^* \omega_0 \omega_1}{m \omega_m L} e^{-i\Delta\omega t}, \quad (2)$$

where $\Delta\omega = \omega_0 - \omega_1 - \omega_m$ is the possible detuning. Remind that we assume D_0 as a constant.

One can find the solutions of (1, 2) in the following form $D_1(t) = D_1 e^{(\lambda - \Delta\omega/2)t}$, $X^*(t) = X^* e^{(\lambda + \Delta\omega/2)t}$ and write down the characteristic equation. The parametric oscillatory instability will appear if real part of one of the characteristic equation roots is positive.

In LIGO design the values of δ_0 and δ_1 are of the order of the bandwidth the gravitational burst spectrum is expected to lie in, i.e. $\simeq 2\pi \times 100 \text{ s}^{-1}$. On the other hand many efforts were made to reduce the value of δ_m to the lowest possible level and thus to decrease the threshold of sensitivity caused by Brownian noise. In existing today fused silica mirrors $Q_m \simeq 10^6 - 2 \times 10^7$ and even for $\omega_m = 10^7 \text{ s}^{-1}$ the value of $\delta_m \leq 10 \text{ s}^{-1}$. Thus we can assume that $\delta_m \ll \delta_1$ and obtain the instability condition in simple form:

$$\frac{\mathcal{R}_0}{\left(1 + \frac{\Delta\omega^2}{\delta_1^2}\right)} > 1, \quad (3)$$

$$\mathcal{R}_0 = \frac{\mathcal{E}_0}{2mL^2\omega_m^2} \frac{\omega_1\omega_m}{\delta_1\delta_m} = \frac{2\mathcal{E}_0Q_1Q_m}{mL^2\omega_m^2}. \quad (4)$$

For estimates we assume parameters corresponding to LIGO-II to be:

$$\begin{aligned} \omega_m &= 2 \times 10^5 \text{ sec}^{-1}, & \delta_m &= 5 \times 10^{-3} \text{ sec}^{-1}, \\ \delta_1 &= 6 \times 10^2 \text{ sec}^{-1}, & \omega_1 &\simeq 2 \times 10^{15} \text{ sec}^{-1}, \\ \mathcal{E}_0 &\simeq 3 \times 10^8 \text{ erg}, & L &= 4 \times 10^5 \text{ cm} \\ m &= 10^4 \text{ g}, \end{aligned} \quad (5)$$

The mechanical frequency ω_m is about the frequency of the lowest mirror elastic (longitudinal or drum) mode and it has the same order as the inter-modal interval $\sim \pi c/L \simeq 2 \times 10^5 \text{ sec}^{-1}$ between optical modes of FP resonator. The mechanical relaxation rate δ_m corresponds to the loss angle $\phi \simeq 5 \times 10^{-8}$ (quality factor $Q_m \simeq 2 \times 10^7$) for fused silica. The value of energy \mathcal{E}_0 corresponds to the value of circulating power about $W \simeq c\mathcal{E}_0/2L \simeq 10^{13} \text{ erg/s} = 10^6 \text{ Watt}$.

For these parameters we have obtained the estimate of coefficient \mathcal{R}_0 for the resonance case ($|\Delta\omega| \ll \delta_1$):

$$\mathcal{R}_0 \simeq 300 \gg 1$$

It means that the critical value of stored energy \mathcal{E}_0 for the instability initiation will be 300 times smaller than the planned value $\simeq 3 \times 10^8 \text{ erg} = 30 \text{ J}^1$.

For nonresonance case and planned value of \mathcal{E}_0 the "borders" of detuning $\Delta\omega_{crit}$ within the system is unstable, are relatively large: $\Delta\omega_{crit} = \delta_1\sqrt{\mathcal{R}_0} \simeq 1.7 \times 10^4 \text{ sec}^{-1}$.

III. CONSIDERATIONS OF THREE-DIMENSIONAL MODES ANALYSIS

The numerical estimates for the values of factor \mathcal{R}_0 and detuning $\Delta\omega$ obtained in the preceding section

¹ It is worth to note that if sapphire is chosen then due to the larger value of Q_m the factor \mathcal{R}_0 will be even bigger: $\mathcal{R}_0 \simeq 5 \times 10^3$. It is another argument which is not in favor of this material for mirrors.

have to be regarded as some kind of warning about the reality of the undesirable parametric instability effect. In the simplified analysis we have ignored the nonuniform distribution of optical fields and of mechanical displacements over the mirror's surface. It is evident that more accurate analysis has to be done. Below we present several considerations about further necessary analysis.

A. The frequency range of "dangerous" optical and elastic modes

The values of the mirror's radius R and thickness H for LIGO-II are not yet finally defined. Due to the necessity to decrease the level of thermoelastic and thermorefractive noises [4,5,9,10] the size of the light spot on the mirror's surface is likely to be substantially larger than in LIGO-I and the light density distribution in the spot is not likely to be a gaussian one (to evade substantial diffractive losses) [10]. Thus the presented below estimates for gaussian optical modes may be regarded only as the first approximation in which the use of analytical calculations is still possible.

The resonance conditions $|\omega_0 - \omega_1 - \omega_m| < \delta_1$ may be obtained with a relatively high probability for many optical Stokes and mirror elastic modes combinations. If we assume the main optical mode to be gaussian one with waist radius w_0 of the caustic (the optical field amplitude distribution in the middle between the mirrors is $\sim e^{-r^2/w_0^2}$), and if we assume also that the Stokes mode may be described by generalized Laguerre functions (Gauss-Laguerre beams) then the set of frequency distances $\Delta\omega_{opt}$ between the main and Stokes modes is determined by three integer numbers:

$$\Delta\omega_{opt} \simeq \frac{\pi c}{L} \left(K + \frac{2(2N+M)}{\pi} \arctan \frac{L\lambda_0}{2\pi w_0^2} \right), \quad (6)$$

where λ_0 is the wave length, $K = 0 \pm 1, \pm 2 \dots$ is the longitudinal index, $N = 0, 1, 2 \dots$, and $M = 0, 1, 2 \dots$ are the radial and angular indices.

For $w_0 \simeq 5.9 \text{ cm}$ the beam radius on the mirror's surface is equal to $w \simeq 6 \text{ cm}$, corresponding to the level of diffractive losses about 20 ppm for mirror radius of $R = 14 \text{ cm}$. In this case the equation (6) has the following form:

$$\Delta\omega_{opt} \simeq (2.4 K + 0.56 N + 0.28 M) \times 10^5 \text{ s}^{-1}. \quad (7)$$

We see that the distance between optical modes is not so large, i.e. $\simeq 3 \times 10^4 \text{ s}^{-1}$. In units of optical modes bandwidth $2\delta_1 \simeq 10^3 \text{ s}^{-1}$ it is about $3 \times 10^4 / 2\delta_1 \simeq 30$. Thus assuming that the value of elastic mode frequency can be an arbitrary one we can roughly estimate the probability that the resonance condition is fulfilled as $\sim 1/30$.

The order of the distance $\Delta\omega_{m0}$ between the frequencies for the first several elastic modes is about

$\Delta\omega_{m0} \simeq \pi v_s/d \simeq 2 \times 10^5 \text{ s}^{-1}$ (d is the dimension of the mirror and v_s is the sound velocity). It is about one order larger than the distance between the optical modes. However, for higher frequencies ω_m these intervals become smaller and can be estimated by formula

$$\Delta\omega_m \simeq \frac{\pi\Delta\omega_{m0}^3}{2\omega_m^2}$$

Even for $\omega_m \simeq 6 \times 10^5 \text{ s}^{-1}$ the intervals between the elastic and optical modes become equal to each other and has value about $\simeq 3 \times 10^4 \text{ s}^{-1}$. And for $\omega_m \sim 10^7 \text{ s}^{-1}$ the distances between elastic modes become of the order of optical bandwidth $2\delta_1$. Therefore, the resonance condition for these frequencies is practically always fulfilled.

On the other hand according to (4) the factor \mathcal{R}_0 decreases for higher elastic frequencies ω_m . In addition the loss angle in fused silica usually slightly increases for higher frequencies [11,12]. Assuming that the upper value $\omega_m \simeq 2 \times 10^6 \text{ s}^{-1}$, $Q_m \simeq 3 \times 10^6$ and other parameters correspond to (5) we obtain $\mathcal{R}_0 \simeq 1$. Therefore, the elastic modes which "deserve" accurate calculations lie within the range between several tens and several hundreds kiloHertz. The total number of these modes is about several hundreds.

B. The Matching between the Mechanical Displacements and Light Density Distributions

The simplified model described in section II is approximately valid for the uniform over all the mirror's surface distribution of optical field density and pure longitudinal elastic mode. The equations for this model can be extended for any distributions of mechanical displacements in the chosen elastic mode and for any distribution of the light field density in chosen optical modes. This extension can be done by adding a dimensionless factor Λ in (3):

$$\frac{\mathcal{R}_0\Lambda}{\left(1 + \frac{\delta\omega^2}{\delta_1^2}\right)} > 1, \quad (8)$$

$$\Lambda = \frac{V \left(\int f_0(\vec{r}_\perp) f_1(\vec{r}_\perp) u_z d\vec{r}_\perp \right)^2}{\int |f_0|^2 d\vec{r}_\perp \int |f_1|^2 d\vec{r}_\perp \int |\vec{u}|^2 dV}. \quad (9)$$

Here f_0 and f_1 are the functions of the distributions over the mirror's surface of the optical fields in the main and Stokes optical modes correspondingly, vector \vec{u} is the spatial vector of displacements in elastic mode, u_z is the component of \vec{u} , normal to the mirror's surface, $\int d\vec{r}_\perp$ corresponds to the integration over the mirror's surface and $\int dV$ — over the mirror's volume V .

It is necessary to know the functions f_0 , f_1 , \vec{u} in order to calculate the factors Λ for different mode combinations. But to the best of our knowledge there is no analytical form for \vec{u} in the case of cylinder with

free boundary conditions [11]. Our approximate estimates show that there is substantial number of modes combinations for which factor Λ is large enough to satisfy the condition (8) for parameters (5). We do not present the details of these estimates here because they are rather rough. It is evident that a complete numerical analysis which includes nongaussian distributions of optical fields is necessary to do.

IV. CONCLUSION

The simplified model analysis of parametric oscillatory instability and considerations about the real model presented above may be regarded only as the first step along the route to obtain a guarantee to evade this undesirable effect. Summing up we may formulate several recommendations for the next steps:

1. Due to the finite size of the mirror and to the use of the nongaussian distribution of light density we think that the accurate numerical analysis of different optical and elastic mode combinations (candidates for the parametric instability) is inevitably necessary. This problem (numerical calculations for the elastic modes) has been already solved partially [13].
2. In the same time the numerical analysis may not give an absolute guarantee because the fused silica pins and fibers will be attached to the mirror. This attachment will change the elastic modes frequency values (and may be also the distribution). In addition the unknown Young modulus and fused silica density inhomogeneity will limit the numerical analysis accuracy. Thus the direct measurements for several hundreds of probe mass elastic modes eigenfrequencies values and quality factors are also necessary.
3. When more "dangerous" candidates of elastic and Stokes modes will be known, their undesirable influence can be depressed. For example it can be done by small the change of mirror's shape.
4. It is also reasonable to perform direct tests of the optical field behavior with smooth increase of the input optical power: it will be possible to register the appearance of the photons at the Stokes modes and the rise of the Q_m in the corresponding elastic mode until the power W in the main optical mode is below the critical value.
5. Apart from above presented case of the oscillatory instability it is likely that there are similar instability in which other mechanical modes are involved (especially violin ones which also have eigen frequencies several tens kHz and higher). There are also additional instability for the pendulum mode in the mirror's suspension (in the

case of small detuning of pumping optical frequency out of resonance). These potential "dangers" also deserve accurate analysis.

We think that the parametric oscillatory instability effect can be excluded in the laser gravitational antennae after this detailed investigation.

ACKNOWLEDGEMENTS

Authors are very grateful to H. J. Kimble, S. Witcomb and especially to F. Ya. Khalili for help, stimulating discussions and advises. This work was supported in part by NSF and Caltech grants and by Russian Ministry of Industry and Science and Russian Foundation of Basic Researches.

APPENDIX A: LAGRANGIAN APPROACH

Let us denote $q_0(t)$ and $q_1(t)$ as generalized coordinates for the FP resonator optical modes with frequencies ω_0 and ω_1 correspondingly, so that their vector potentials (A_0 , A_1), electrical (E_0 , E_1) and magnetic (H_0 , H_1) fields are the following:

$$\begin{aligned} A_i(t) &= \sqrt{\frac{2\pi c^2}{S_i L}} (f_i e^{ik_i z} - f_i^* e^{-ik_i z}) q_i(t), \\ E_i(t) &= -\sqrt{\frac{2\pi}{S_i L}} (f_i e^{ik_i z} - f_i^* e^{-ik_i z}) \partial_t q_i(t), \\ H_i(t) &= \sqrt{\frac{2\pi}{S_i L}} (f_i e^{ik_i z} + f_i^* e^{-ik_i z}) \omega_i q_i(t), \\ f_i &= f_i(\vec{r}_\perp, z), \quad S_i = \int |f_i|^2 d\vec{r}_\perp. \end{aligned}$$

Let also denote $x(t)$ as generalized coordinate of the considered elastic oscillations mode with displacement spatial distribution described by the vector $\vec{u}(\vec{r})$. Now we can write down the lagrangian:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_m + \mathcal{L}_{int}, \\ \mathcal{L}_0 &= \int \frac{L(\langle E_0 \rangle^2 - \langle H_0 \rangle^2)}{8\pi} d\vec{r}_\perp = \frac{\partial_t q_0^2}{2} - \frac{\omega_0^2 q_0^2}{2}, \\ \mathcal{L}_1 &= \frac{\partial_t q_1^2}{2} - \frac{\omega_1^2 q_1^2}{2}, \\ \mathcal{L}_m &= \frac{M(\partial_t x)^2}{2} - \frac{M\omega_m^2 x^2}{2}, \\ M &= \rho \int_V |\vec{u}(\vec{r})|^2 dV, \\ \mathcal{L}_{int} &= - \int \frac{x u_z \langle H_0 + H_1 \rangle^2}{8\pi} \Big|_{z=0} d\vec{r}_\perp = \\ &= -2\omega_0 \omega_1 q_0 q_1 B \frac{x}{L}, \\ B &= \frac{\int f_0(\vec{r}_\perp) f_1(\vec{r}_\perp) u_z d\vec{r}_\perp}{\sqrt{\int |f_0|^2 d\vec{r}_\perp \int |f_1|^2 d\vec{r}_\perp}} \end{aligned}$$

We consider only one mechanical mode below. Now we can write down the equations of motion (adding losses in each degree of freedom):

$$\begin{aligned} \partial_t^2 q_0 + 2\delta_0 \partial_t q_0 + \omega_0^2 q_0 &= -B \frac{2x}{L} \omega_0 \omega_1 q_1, \\ \partial_t^2 q_1 + 2\delta_1 \partial_t q_1 + \omega_1^2 q_1 &= -B \frac{2x}{L} \omega_1 \omega_0 q_0, \\ \partial_t^2 x + 2\delta_m \partial_t x + \omega_m^2 x &= -B \frac{2\omega_0 \omega_1}{ML} q_0 q_1. \end{aligned}$$

Introducing slowly varying amplitudes we can rewrite these equations as:

$$\begin{aligned} q_0(t) &= D_0(t) e^{-i\omega_0 t} + D_0^*(t) e^{i\omega_0 t}, \\ q_1(t) &= D_1(t) e^{-i\omega_1 t} + D_1^*(t) e^{i\omega_1 t}, \\ x(t) &= X(t) e^{-i\omega_m t} + X^*(t) e^{i\omega_m t}, \\ \Delta\omega &= \omega_0 - \omega_1 - \omega_m, \\ \partial_t D_0 + \delta_0 D_0 &= \frac{iBX D_1 \omega_1}{L} e^{i\Delta\omega t}, \\ \partial_t D_1 + \delta_1 D_1 &= \frac{iBX^* D_0 \omega_0}{L} e^{-i\Delta\omega t}, \\ \partial_t X + \delta_m X &= \frac{iBD_0 D_1^* \omega_0 \omega_1}{M\omega_m L} e^{-i\Delta\omega t}, \end{aligned} \quad (A1)$$

$$(A2)$$

We can see that this system (A2, A1) coincides with (2, 1) if $B = 1$, and $M = m$.

For the simplest resonance case $\delta\omega = 0$ it is easy to substitute (A2) into (A1) and to obtain the condition of parametric instability (in the frequency domain):

$$\begin{aligned} D_1(\delta_1 - i\Omega) &= \frac{iBD_0\omega_0}{L} \times \frac{-iBD_0^* D_1 \omega_0 \omega_1}{M\omega_m L(\delta_m + i\Omega)}, \\ \text{Condition of instability:} \\ 1 &< \frac{B^2 |D_0|^2 \omega_0^2 \omega_1^2}{M\omega_m L^2 \delta_1 \delta_m} \end{aligned}$$

Now we can express the energy \mathcal{E}_0 in mode "0" in terms of $|D_0|^2$:

$$\begin{aligned} \mathcal{E}_0 &= \frac{\partial_t q_0^2}{2} + \frac{\omega_0^2 q_0^2}{2} = \\ &= \frac{1}{2} \left((-i\omega_0)^2 [D_0 e^{-i\omega_0 t} - D_0^* e^{i\omega_0 t}]^2 + \right. \\ &\quad \left. + \omega_0^2 [D_0 e^{-i\omega_0 t} + D_0^* e^{i\omega_0 t}]^2 \right) = \\ &= 2\omega_0^2 |D_0|^2. \end{aligned}$$

Now we can write down the condition of parametric instability in the following form:

$$\frac{\mathcal{E}_0 B}{2m\omega_m^2 L^2} \times \frac{\omega_1 \omega_m}{\delta_1 \delta_m} > 1, \quad (A3)$$

$$\Lambda = \frac{B^2 m}{M} = \frac{V \left(\int f_0(\vec{r}_\perp) f_1(\vec{r}_\perp) u_z d\vec{r}_\perp \right)^2}{\int |f_0|^2 d\vec{r}_\perp \int |f_1|^2 d\vec{r}_\perp \int |\vec{u}|^2 dV} \quad (A4)$$

which accurately coincides with (8, 9).

Let us deduce the instability condition for nonresonance case. We are looking for the solution of (1, 2) in the following form:

$$D_1(t) = D_1 e^{\lambda_- t}, \quad X^*(t) = X^* e^{\lambda_+ t},$$

$$\lambda_- = \lambda - \frac{i\Delta\omega}{2}, \quad \lambda_+ = \lambda + \frac{i\Delta\omega}{2},$$

and writing down the characteristic equation as:

$$(\lambda_+ + \delta_1)(\lambda_- + \delta_m) - A = 0, \quad \frac{D_0^2 \omega_0^2 \omega_1 \Lambda}{m \omega_m L^2} = A.$$

The solutions of characteristic equation are:

$$\lambda_{1,2} = -\frac{\delta_1 + \delta_m}{2} \pm \sqrt{\text{Det}},$$

$$\text{Det} = \left(\frac{\delta_1 - \delta_m}{2} - \frac{i\Delta\omega}{2} \right)^2 + A.$$

The condition of instability is the following:

$$\Re \sqrt{\text{Det}} > \frac{\delta_1 + \delta_m}{2}. \quad (\text{A5})$$

Using a convenient formula:

$$\text{Det} = a + ib,$$

$$\Re \sqrt{\text{Det}} = \frac{\sqrt{2}}{2} \sqrt{\sqrt{a^2 + b^2} + a}.$$

we can rewrite the condition (A5) as:

$$\frac{1}{2} \left(\sqrt{a^2 + b^2} + a \right) > \left(\frac{\delta_1 + \delta_m}{2} \right)^2, \quad (\text{A6})$$

$$a^2 + b^2 = A^2 + \left(\frac{(\delta_1 - \delta_m)^2}{4} + \frac{\Delta\omega^2}{4} \right)^2 +$$

$$+ 2A \left(\frac{(\delta_1 - \delta_m)^2}{4} - \frac{\Delta\omega^2}{4} \right) \quad (\text{A7})$$

Note that for the resonance case ($\Delta\omega = 0$) the solution of (A5 or A6) is known: $A > \delta_1 \delta_m$. For our case $\delta_m \ll \delta_1$ it means that $A \ll \delta_1^2$. Therefore for small detuning $\Delta\omega \ll \delta_1$ we can expand $a^2 + b^2$ in series in terms of A and rewrite condition (A6) as:

$$\frac{1}{2} \left(\sqrt{a^2 + b^2} + a \right) \simeq \frac{A}{2} + \frac{(\delta_1 - \delta_m)^2}{4} +$$

$$+ \frac{A}{2} \frac{\frac{(\delta_1 - \delta_m)^2}{4} - \frac{\Delta\omega^2}{4}}{\frac{(\delta_1 - \delta_m)^2}{4} + \frac{\Delta\omega^2}{4}}$$

$$A \frac{\frac{(\delta_1 - \delta_m)^2}{4}}{\frac{(\delta_1 - \delta_m)^2}{4} + \frac{\Delta\omega^2}{4}} > \delta_1 \delta_m,$$

$$\text{Or } A > \delta_1 \delta_m \times \frac{(\delta_1 - \delta_m)^2 + \Delta\omega^2}{(\delta_1 - \delta_m)^2}. \quad (\text{A8})$$

Let us underline that condition (A8) is obtained for small detuning $\Delta\omega \ll \delta_1$. However, considering situation more attentively one can conclude that expansion in series (and consequently the formula (A8)) is valid for the condition:

$$A \ll \frac{(\delta_1 - \delta_m)^2}{4} + \frac{\Delta\omega^2}{4} \quad (\text{A9})$$

We see that this condition is fulfilled for the solution (A8). Therefore we conclude that solution (A8) is approximately valid for any detunings $\Delta\omega$.

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